A Brief Survey of Key Topics in Group Theory

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1. Basics of Group Theory
   - Definitions
   - What is a group?
   - Isomorphisms and Homomorphisms
   - Commutativity
   - Order of a Group
   - Order of an Element

2. Symmetry

3. Subgroups, Groups, and Cosets

4. Applications

Before introducing groups, we must first understand binary operations.

Definition (Binary Operation)

If G is a nonempty set, a binary operation μ on G is a function $\mu : G \times G \to G$.

Definition (Associativity)

A binary operation $*$ on set G is associative if

$$(a * b) * c = a * (b * c)$$

for all $a, b, c \in G$.

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Examples of Binary Operations

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2. Matrix multiplication.
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Definition (Group)

A group $(G, *)$ is a set G with binary operation $* : G \times G \to G$ that satisfies:

1. *Closure:* $g * h \in G$. We say that G is closed under $*$.
2. *Associativity:* for any $g, h, i \in G$ we have $(g * h) * i = g * (h * i)$.
3. *Identity:* $e * g = g * e = g$ for all $g \in G$.
4. *Inverse:* Every element $g \in G$ has an inverse g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

Examples

Some examples include:

1. The integers under addition, subtraction, and multiplication.
2. The set $GL_2(\mathbb{R})$ of 2 by 2 invertible matrices over the reals with matrix multiplication as the binary operation.
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Proof that the Identity is Unique

Proof.

Let there be group G such that $a \in G$ and e_1 and e_2 are both identity elements of G. Then:

$$a^{-1} * a = e_1$$

$$a^{-1} * a = e_2$$

$$e_1 = a^{-1} * a = e_2$$

$$e_1 = e_2$$

$\square$

Isomorphisms

Definition (Isomorphism)

Two groups $(G, *)$ and $(H, \circ)$ are said to be isomorphic if there is a one-to-one correspondence $\theta : H \to G$ such that

$$\theta(g_1 * g_2) = \theta(g_1) \circ \theta(g_2)$$

for all $g_1, g_2 \in G$. The mapping θ is called an isomorphism and we say that G is isomorphic to H (written as $G \cong H$).

Homomorphisms

Definition (Homomorphism)

If θ satisfies the previously mentioned property but is not a one-to-one correspondence, we say θ is homomorphism.

Definition (Commute)

If G is a group and $g, h \in G$, if $gh = hg$ we say that g and h commute.

Definition (Abelian)

$g * h = h * g$ for all $g, h \in G$, then we say G is an abelian group. Some examples include:

- The group of integers under addition.
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Order of an Element

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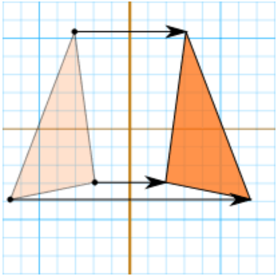
The order of an element $g \in G$, written as $o(g)$, is the smallest natural number n, such that $g^n = e$. If no n exists we say the element has an infinite order.

Definition (Symmetry)

A symmetry is a transformation of a figure under which the figure is invariant.



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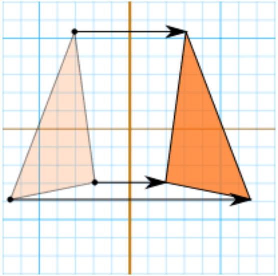

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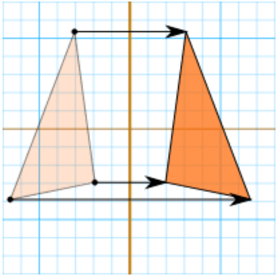

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Proof of the 5th Quadratic

Theorem (Abel-Ruffini Theorem)

There is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients.

- Uses roots of unity
- Requires proof that the symmetric group of the 5th polynomial is not solvable and that there are polynomials with symmetric Galois groups.

$$P(x) = x^n + b_1 x^{n-1} + b_{n-1}x + b_n = (x - x_1)\cdots(x - x_n)$$

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Wallpaper Groups

Definition (Fundamental Region, Wallpaper group)

A Fundamental Region is a pattern or region that repeats in many directions.
A Wallpaper Group is the group of symmetries of a fundamental region, under the binary operation of composition.

Wallpaper groups include rotations, reflections, translations, and glide reflections. With these transformations, there are 17 possible wallpaper groups in $\mathbb{R}^2$

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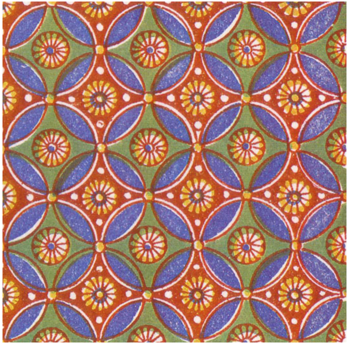
Wallpaper Group Example



Definition (Subgroup, Proper Subgroup)

A subset $H \subseteq G$ is a subgroup of G if

- H is not empty.
- If $h, k \in H$ then $hk \in H$
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We write $H \leq G$ if H is a subgroup of G. If also $H \neq G$, we say that H is a proper subgroup and write $H < G$.

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Lattices of subgroups

Definition (Lattice)

The lattice of subgroups of a group G is the graph where subgroups are nodes.
To construct the lattices of a finite subgroup:

- Plot subgroups starting at the bottom of the lattice with identity subgroup $\{1\}$.
- Plot subgroups of larger order progressively higher in the lattice.
- End at the top of the lattice with G.
- Draw a line upwards from A to B if $A \leq B$ and no subgroups exist properly between A and B.

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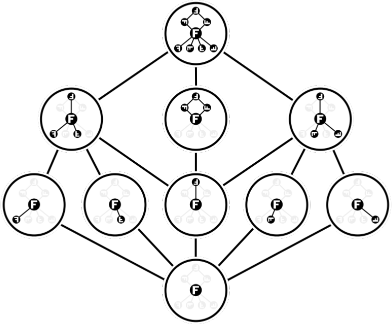
The Dihedral group of order 4 has ten subgroups:



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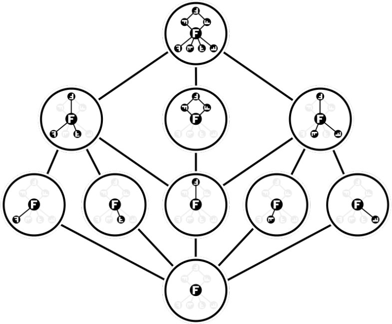

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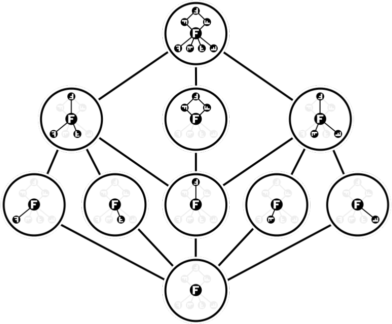

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Cyclic Groups

Definition (Cyclic)

A group H is cyclic if it can be generated by a single element, i.e, there is some element $x \in H$ such that

$$H = \{x^n | n \in \mathbb{Z}\},$$

where as usual the operation is multiplication.

If using additive notation, H is cyclic if it can be written as

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We denote the fact that H is generated by x as:

$$H = \langle x \rangle$$

A cyclic group can have more than one generator.

Example

We can also write $H = \langle x^{-1} \rangle$ since both n and $-n$ run over all integers and

$$(x^{-1})^n = x^{-n}$$

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Order of Cyclic Groups

Proposition

If $H = \langle x \rangle$, then $|H| = |x|$.

- *if $|H| = n < \infty$, then $x^n = 1$ and $1, x, x^2, \ldots, x^{n-1}$ are all distinct elements of H.*
- *if $|H| = \infty$, then $x^n \neq 1$ for all $n \neq 0$ and $x^a \neq x^b$ for all $a \neq b$ in $\mathbb{Z}$.*

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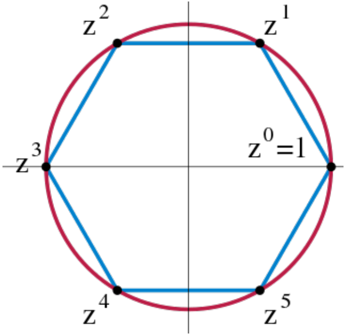

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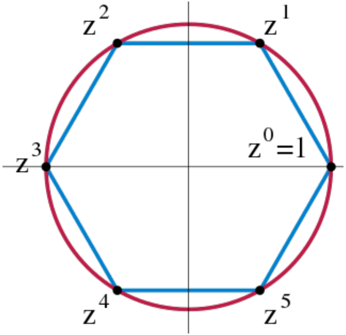

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Dihedral Groups

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This includes rotations and reflections.

For a n-gon, the algebraic way of representing this group is D_{2n} and the geometric way is D_n.

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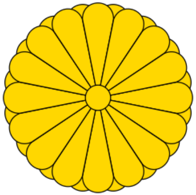
This is a 2D example of the dihedral group of a 16-gon.



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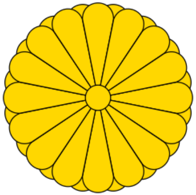

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Problem (Burnside's Problem)

Must a finitely generated group in which every element has finite order necessarily be finite group?

The existence of a Tarski group for a prime p provides a negative answer to the Burnside problem for that prime p.
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Lagrange's Theorem

Lagranges Theorem is one of the central theorems of Abstract Algebra and its proof uses several important ideas. Let's look at Lagrange's Theorem and then try to prove it.

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Lagrange's Theorem Proof

Before we prove Lagrange's Theorem, we need to look at some lemmas.

Lemma

If $H \leq G$ there is a one-to-one correspondence between H in any coset of H.

Lemma

If $H \leq G$, then the left coset relation, $g_1 \sim g_2$ if $g_1 H = g_2 H$ is an equivalence relation.

Lemma

Let S be a set and $\sim$ be an equivalence relation on S. If A and B are two equivalence classes with $A \cap B \neq \emptyset$, then $A = B$.

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Before we prove Lagrange's Theorem, we need to look at some lemmas.

Lemma

If $H \leq G$ there is a one-to-one correspondence between H in any coset of H.

Lemma

If $H \leq G$, then the left coset relation, $g_1 \sim g_2$ if $g_1 H = g_2 H$ is an equivalence relation.

Lemma

Let S be a set and $\sim$ be an equivalence relation on S. If A and B are two equivalence classes with $A \cap B \neq \emptyset$, then $A = B$.

Lagrange's Theorem Proof

With these lemmas in mind, we can prove Lagrange's Theorem.

Proof.

Let $\sim$ be the left coset equivalence relation we defined in the second lemma. The last lemma states that any two distinct cosets of $\sim$ are disjoint. This means we can say

$$G = (g_1 H) \cup (g_2 H) \cup \ldots \cup (g_n H)$$

The first lemma shows that the order of each coset is the same as the order of H, so

$$|G| = |g_1 H| + |g_2 H| + \ldots + |g_n H| = n|H|$$

$$|G| = n|H|$$

showing that $|G|$ is divisible by $|H|$. □

Extension to Number Theory

Theorem (Fermat's Little Theorem)

Let p be a prime number and let a be an integer not divisible by p. Then,

$$a^p \equiv a \pmod{p}.$$

Proof.

Let set $G = (1, 2, \cdots, p-1)$ form a group with the operation of multiplication. Let us assume that a is in the range $1 \geq a \geq p-1$, that, a is an element of G. Let k be of the order of a, that is, k is the smallest positive integer such that $a^k \equiv 1 \pmod{p}$. Then the numbers $1, a, a^2, \cdots, a^{k-1}$ reduced modulo p form a subgroup of G whose order is k and therefore by Lagrange's theorem, k divides the order of G, which is $p-1$. $\square$

Extension to Geometry

Theorem

There are only three polygons that can tile the plane: equilateral triangles, squares, and regular hexagons.

Proof.

Let G be the symmetry group of the tiling, which must contain at least one n-fold rotational symmetry. If G has any reflections, then the vertex figure of the tiling must be a regular polygon of $2-n$ sides. This is impossible. Therefore, G can only consist of rotations, which implies that the angles around each vertex must sum to a multiple of $\frac{360}{n}$ degrees. This is only possible for $n = 3, 4, 6$ which correspond to the 3 regular tilings that we identified. $\square$

References

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Thank you! Any Questions?